# Sums of $k$-th Powers and Fourier Coefficients of <br> Cusp Forms ${ }^{1}$ 

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#### Abstract

In this paper, we will first establish a power saving result for the shifted convolution sums of $k$-th powers and the normalized Fourier coefficients of $\mathrm{SL}_{2}(\mathbb{Z})$ cusp forms. Later we will generalize the result to higher rank cases.


Keywords Shifted convolution sums, cusp forms, circle method, Voronoi formula.
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## 1 Introduction

The study of shifted convolution sums of arithmetic functions is a classical theme in analytic number theory. In 1927, Ingham [Ing] first obtained the following asymptotic formula for the divisor function $d(n)$ :

$$
\sum_{n \leq X} d(n+1) d(n) \sim \frac{6}{\pi^{2}} X(\log X)^{2}
$$

as $X \rightarrow \infty$. Later, Deshouillers and Iwaniec [DI] showed, for any $\epsilon>0$,

$$
\sum_{n \leq X} d(n+1) d(n) \sim \frac{6}{\pi^{2}} X(\log X)^{2}+c_{1} X \log X+c_{2} X+O\left(X^{2 / 3+\epsilon}\right)
$$

where $c_{1}, c_{2}$ are some constants not related to $\epsilon$.
Since Selberg's paper [Sel], the shifted convolution of $\mathrm{GL}_{2}$ Fourier coefficients has received much attention. Indeed, nontrivial bounds of $\mathrm{GL}_{2} \times \mathrm{GL}_{2}$ shifted convolutions have profound implications on subconvexity problems and quantum unique ergodicity.

[^0]([Blo1], [Blo2] and [Hol]). For shifted convolution of higher rank cases, one can refer to [ Pitt$]$ and [ Mu .

This paper will investigate the shifted convolution of Fourier coefficients of cusp forms and $k$-th powers. We can formulate the problem as follows: let $s$ and $k$ be natural numbers. Denote by $r_{s, k}(n)$ the number of representations of a positive integer $n$ as the sum of $s$ positive integral $k$-th powers. Additionally, denote by $a_{f}(n)$ the $n$-th normalized Fourier coefficient of a holomorphic cusp form of weight $l$,

$$
f(z)=\sum_{n=1}^{\infty} a_{f}(n) n^{\frac{l-1}{2}} e(n z) \in \mathcal{S}_{l}\left(\mathrm{SL}_{2}(\mathbb{Z})\right)
$$

or a Maass cusp form

$$
f(z)=y^{1 / 2} \sum_{n \neq 0} a_{f}(n) K_{i t}(2 \pi|n| y) e(n x)
$$

with the Laplacian eigenvalue $\lambda=\frac{1}{4}+t^{2}$. Let $\phi(x)$ be a smooth function compactly supported in $[1 / 2,1]$. In this paper, we are interested in the following (smooth) shifted convolution sum

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{f}(n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \tag{1.1}
\end{equation*}
$$

By Deligne's seminal work [Del] on the Fourier coefficients of holomorphic cusp forms, the trivial bound for the shifted convolution is $X^{\frac{s}{k}}+\epsilon$ for any $\epsilon>0$. On the other hand, by the cancellations in the sum of $a_{f}(n)$ twisted by additive characters, it is natural to ask whether we can establish power savings for such shifted convolutions. For the nonholomorphic case, we have not proved the Ramanujan conjecture yet. However, it is also of great interest to establish similar power saving results for the Maass form case.

More generally, let $a_{\pi}\left(n_{1}, \ldots, n_{m-1}\right)$ denote a Whittaker-Fourier coefficient of a cusp form $\pi$ on $\mathrm{SL}_{m}(\mathbb{Z})$. We can establish a shifted convolution of the form

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \tag{1.2}
\end{equation*}
$$

and expect power saving results for such shifted convolutions.
The goal of this paper is to establish a non-trivial bound for 1.1 and 1.2 of the type $O\left(X^{\frac{s}{k}-\delta+\epsilon}\right)$. Here $\delta$ is some positive number (depending on $s, k$ ) and $\epsilon$ can be any positive number.

Here are many papers investigating the case $k=2$. In [Luo2], Luo derived a Voronoitype formula for $r_{s, 2}(n)$ and obtained that, for $s \geq 2$ and $f$ a fixed holomorphic or Maass cusp form,

$$
\sum_{n \leq X} a_{f}(n+1) r_{s, 2}(n) \ll X^{\frac{s}{2}-\delta_{s}+\epsilon}
$$

for some $\delta_{s}>0$ depending on $s$ and any $\epsilon>0$.

For the higher rank case, Sun [Sun] used the GL(3) Voronoi formula and proved

$$
\sum_{n=1}^{\infty} a_{\pi}(1, n+b) r_{3,2}(n) \ll X^{\frac{3}{2}-\frac{1}{8}+\epsilon}
$$

for any $\epsilon>0$ and $b$ an integer satisfying $1 \leq b \leq X$. Later Jiang and Lü generalized this result to GL( $m$ ) Hecke-Maass cusp forms [JL].

However, establishing a nontrivial bound for $k \geq 3$ can be much more challenging. Luo [Luo1] first considered the case $k \geq 3$ : for GL(2) case, he obtained the power saving results for $s \geq k+1$ provided that $k=3$ or 4 . Additionally, he established the power saving for large $k$.

In section 3, we will prove the following theorem:
Theorem 1.1. Let $k \geq 3$. Define $A(k)$ to be

$$
A(k)= \begin{cases}\frac{(k+1)^{2}}{4} & \text { if } k \text { is odd } \\ \frac{k^{2}+k}{4} & \text { if } k \text { is even }\end{cases}
$$

Then for $s>A(k)$, we have

$$
\sum_{n=1}^{\infty} a_{f}(n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{k}-\delta+\epsilon}
$$

for some $\delta>0$ depending on $s, k$ and any $\epsilon>0$.
Remark 1.2. In Luo's paper [Luol], Theorem 1 established the power saving for $s \geq$ $k^{2}+O(k)$ when $f$ is a GL(2) cusp form. Theorem 1.1 improves his result by reducing the number of variables to $\frac{1}{4} k^{2}+O(k)$.

The proof makes use of the circle method. For the major arc, we will apply Voronoi's formula to establish a power saving. For the minor arc, we can obtain the power saving using Bourgain's improvement on Hua's inequality.

For the higher rank case, we will establish the following theorem in section 4. Before the statement of the theorem, we introduce some notations: denote by $\lceil t\rceil$ the smallest integer no smaller than $t$. Then we define the following functions

$$
f_{1}(k)=\frac{1}{2}\left\{k^{2}+1-\max _{i<k, 2^{i}<k^{2}}\left[\frac{k i-2^{i}}{k-i+1}\right\rceil\right\}=\frac{k^{2}+1}{2}-\frac{1}{2} \frac{\log k}{\log 2}+O(1)
$$

and

$$
f_{2}(k)=\frac{1}{2}\left\{k^{2}+1-\max _{i<k}\left\lceil i \frac{k-i-1}{k-i+1}\right]\right\}=\frac{k^{2}-k}{2}+O(\sqrt{k})
$$

for integers $k \geq 2$. It can be checked that $f_{1}(k) \leq f_{2}(k)$ when $k \leq 12$. For larger $k$, the second expression can be better. The theorem is stated as follows:

Theorem 1.3. Let $\pi$ be an even Maass cusp form on $\mathrm{SL}_{m}(\mathbb{Z})$ with $m \geq 3$. Suppose that $k \geq 3$ and $k \neq 4$ and

$$
s>\min \left\{f_{1}(k), f_{2}(k)\right\},
$$

we have

$$
\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \ll X^{s-\delta+\epsilon}
$$

for some positive $\delta$ and any $\epsilon>0$.
Remark 1.4. For the case $k=4$, we can establish the power saving for $s>8$.
Remark 1.5. In Luo's paper [Luol], Theorem 3 showed the power saving for $s \geq k(k+$ 1) in the higher rank case.Theorem 1.3 improves his result by reducing the number of variables to $\frac{k^{2}-k}{2}+O(\sqrt{k})$.

To prove this, we will again apply the circle method. However, for the case $k=3$, we will use a non-standard circle method.

For simplicity, we would set:

$$
\left.B(k)=\min \left\{f_{1}(k), f_{2}(k)\right\}\right\}
$$

For the case $k=3$, we can prove a stronger result than that in Theorem 1.3. That is, for $s \geq 5$ (Notice that $B(3)=5)$ and $\pi$ being an even Maass cusp form on $\mathrm{SL}_{m}(\mathbb{Z})$, one has

$$
\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) r_{s, 3}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{3}-\frac{1}{4 m}+\epsilon}
$$

for any $\epsilon>0$.
We can improve the power saving when $m=4$ using the standard circle method, which is the following theorem in section 5:

Theorem 1.6. Let $\pi$ be an even Hecke-Maass cusp form on $\mathrm{SL}_{4}(\mathbb{Z})$. Then we have

$$
\sum_{n=1}^{\infty} a_{\pi}(1,1, n+1) \phi\left(\frac{n}{X}\right) r_{5,3}(n) \ll X^{\frac{5}{3}-\frac{1}{12}+\epsilon}
$$

for any $\epsilon>0$.
For the minor arc part, we can apply Hua's inequality. For the major arc, our strategy involves the Voronoi formula of the high rank case and a tight estimation of the exponential sum 5.1.

## 2 Prelimiaries

In the following sections, assume that $s, k, n, l, m$ are positive integers and $X$ is a sufficiently large number. Denote by $f$ a fixed $\mathrm{SL}_{2}(\mathbb{Z})$ cusp form, holomorphic or Maass, and by $a_{f}(n)$ the $n$-th normalized Fourier coefficient of $f(z)$. Suppose that $\pi$ is a fixed even Maass form on $\mathrm{SL}_{m}(\mathbb{Z})$.

For fixed $X>0$, we define the following functions

$$
\begin{aligned}
& \mathcal{F}_{k}(\alpha)=\sum_{m \leq X^{1 / k}} e\left(\alpha m^{k}\right) \\
& \mathcal{G}_{f}(\alpha)=\sum_{n=1}^{\infty} a_{f}(n+1) \phi\left(\frac{n}{X}\right) e(-\alpha n) \\
& \mathcal{G}_{\pi}(\alpha)=\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) \phi\left(\frac{n}{X}\right) e(-\alpha n)
\end{aligned}
$$

In this case, it is clear that

$$
\sum_{n=1}^{\infty} a_{f}(n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right)=\int_{0}^{1} \mathcal{G}_{f}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

and

$$
\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right)=\int_{0}^{1} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

### 2.1 Some Lemmas on $\mathscr{F}_{k}(\alpha)$

$\mathcal{F}_{k}(\alpha)$ is a fundamental object when studying the Waring problem since

$$
r_{s, k}(n)=\int_{0}^{1} \mathcal{F}_{k}^{s}(\alpha) e(-\alpha n) d \alpha
$$

When studying the integral, the circle method plays a central role. Let $P, Q$ be two positive real numbers such that $P Q=X$. Then we define

$$
\mathcal{M}=\bigcup_{1 \leq q \leq P} \bigcup_{\substack{1 \leq a \leq q \\(a, q)=1}} \mathcal{M}(a, q)
$$

where

$$
\mathcal{M}(a, q)=\left\{\alpha:\left|\alpha-\frac{a}{q}\right| \leq \frac{1}{q Q}\right\}
$$

This is a disjoint union once $Q \geq 2 P$. Then set

$$
\mathrm{m}=\left(\frac{1}{Q}, 1+\frac{1}{Q}\right] \backslash \mathcal{M}
$$

$\mathcal{M}$ is called the major arc and $m$ is called the minor arc part. Since $\mathcal{F}_{k}(\alpha)$ is of period 1 , the integral can be written as

$$
r_{s, k}(n)=\int_{0}^{1} \mathcal{F}_{k}^{s}(\alpha) e(-\alpha n) d \alpha=\int_{\mathcal{M}} \mathcal{F}_{k}^{s}(\alpha) e(-\alpha n) d \alpha+\int_{\mathfrak{m}} \mathcal{F}_{k}^{s}(\alpha) e(-\alpha n) d \alpha .
$$

In the proof of Theorem 1.1, we will use the so-called standard choice: for $X>0$, set

$$
P=\frac{X^{1 / k}}{2 k}, \quad Q=\frac{X}{P}
$$

We will use the standard choice to prove Theorem 1.3 except for the extreme case $k=3$. For Theorem 1.6, we will again use the standard choice of $P, Q$.

A similar argument shows:

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{f}(n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right)=\int_{\mathcal{M}} \mathcal{G}_{f}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha+\int_{\mathbf{m}} \mathcal{G}_{f}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha \tag{2.1}
\end{equation*}
$$

Similarly, we also call the integral over $\mathcal{M}$ the major arc and the second integral the minor arc respectively.

We also have a similar expression for $a_{\pi}(1, \ldots, 1, n+1)$ case, that is,

$$
\sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right)=\int_{\mathcal{M}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha+\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

For the major arc part, we have the following theorem, which is Theorem 4.4 in $[\overline{\mathrm{Va}}]$ :
Theorem 2.1 ( $[\mid \mathrm{Va}])$. Suppose that $s \geq \max (5, k+1)$. Then there is a positive number $\delta$ such that whenever $1 \leq n \leq X$,

$$
\int_{\mathcal{M}} \mathcal{F}_{k}^{s}(\alpha) e(-\alpha n) d \alpha=\frac{\Gamma\left(1+\frac{1}{k}\right)^{s}}{\Gamma\left(\frac{s}{k}\right)} \Im_{s, k}(n, P) n^{\frac{s}{k}-1}+O\left(n^{\frac{s}{k}-1-\delta}\right)
$$

where

$$
\Im_{s, k}(n, P)=\sum_{q \leq P} \sum_{\substack{a=1 \\(a, q)=1}}^{q}\left(\frac{1}{q} \sum_{r=1}^{q} e\left(\frac{a r^{k}}{q}\right)\right)^{s} e\left(\frac{-n a}{q}\right)
$$

is called the singular series.
Remark 2.2. By checking the proof of the theorem in Vaughan's book, it is easy to see, when $s=5$ and $k=3, \delta=\frac{1}{12}$.

For simplicity, we set

$$
S(q, a)=\sum_{r=1}^{q} e\left(\frac{a r^{k}}{q}\right)
$$

(For simplicity, the index $k$ is ignored when defining $S(q, a)$.) We have the following lemma for $S(q, a)$, which is Theorem 4.2 in [Va]:

Lemma 2.3 ([|Va]). Suppose that $(a, q)=1$. Then

$$
S(q, a) \ll q^{1-\frac{1}{k}}
$$

Here are also several lemmas for the minor arc. Actually, the following lemma shows that the minor arc of $\mathcal{F}_{k}(\alpha)$ is "small":

Lemma 2.4 (Weyl's Inequality, [Va], [Bo]). Let $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right) \in \mathbb{R}^{k}$. Then we define

$$
f_{k}(\alpha, Y)=\sum_{n \leq Y} e\left(\alpha_{1} n+\cdots+\alpha_{k} n^{k}\right)
$$

Assume that

$$
\left|\alpha_{k}-\frac{a}{q}\right| \leq \frac{1}{q^{2}} \quad(a, q)=1, \quad k \geq 2
$$

Then we have

$$
f_{k}(\alpha, Y) \ll Y^{1+\epsilon}\left(q^{-1}+Y^{-1}+q Y^{-k}\right)^{\sigma(k)}
$$

where $\sigma(k)=\max \left(\frac{1}{k(k-1)}, \frac{1}{2^{k-1}}\right)$. As a direct corollary, for $\alpha \in \mathfrak{m}$, that $|\alpha-a / q|<\frac{1}{q Q}$ implies $q>P$ and hence

$$
\mathcal{F}_{k}(\alpha) \ll Y^{1+\epsilon}\left(q^{-1}+Y^{-1}+q Y^{-k}\right)^{\sigma(k)}
$$

where $Y=X^{1 / k}$.
In addition, for the integral, we have the following Hua's inequality:
Lemma 2.5 (Hua's inequality, [Hua], [Bo]). For fixed positive integer $k$, let $\mathcal{F}_{k}(\alpha)$ be the function defined above, then for $1 \leq l \leq k$

$$
\int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{2^{l}} d \alpha \ll Y^{2^{l}-l+\epsilon}
$$

where $Y=X^{\frac{1}{k}}$. Later Bourgain [Bo] shows that

$$
\int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{l(l+1)} d \alpha \ll Y^{l^{2}+\epsilon}
$$

For $\mathcal{F}_{k}(\alpha)$, another evidence showing that the minor arc is small is the following lemma, which is Theorem 2.1 in (Wo1]:

Lemma 2.6 ([|Wo1]). For the standard choice of the minor arc, we have

$$
\int_{\mathfrak{m}}\left|\mathcal{F}_{k}(\alpha)\right|^{2 s} d \alpha \ll Y^{\frac{1}{2} k(k-1)-1+\epsilon} J_{s, k}(2 Y)
$$

with $Y=X^{1 / k}$ and $J_{s, k}(Y)$ counting the number of solutions to the following homogeneous equations:

$$
\begin{array}{cc}
x_{1}+\cdots+x_{s}= & y_{1}+\cdots+x_{s} \\
x_{1}^{2}+\cdots+x_{s}^{2}= & y_{1}^{2}+\cdots+x_{s}^{2} \\
\cdots & \cdots \\
x_{1}^{k}+\cdots+x_{s}^{k}= & y_{1}^{k}+\cdots+x_{s}^{k}
\end{array}
$$

with $1 \leq x_{i}, y_{i} \leq Y$
Here are some remarks on this result. By checking the proof in [Wo1], we could show, for a general choice of $P, Q$,

$$
\int_{\mathfrak{m}}\left|\mathscr{F}_{k}(\alpha)\right|^{2 s} \ll Y^{\frac{1}{2} k(k-1)+\epsilon} J_{s, k}(2 Y) \max \left\{\frac{1}{Y}, \frac{1}{P}\right\} .
$$

In addition, for all integers $s, k \geq 1$, we have the so-called Vinogradov mean value theorem

$$
J_{s, k}(X) \ll_{s, k, \epsilon} X^{\epsilon}\left(X^{s}+X^{2 s-\frac{k(k+1)}{2}}\right) .
$$

This is proved by [Wo2] for $k=3$ via efficient congruencing and [BDG] for $k \geq 4$ via $l^{2}$-decoupling method. A good reference for Vinogradov mean value theorem is [Pi].

Then combine these results, and we can show: for $s \geq \frac{k(k+1)}{2}$,

$$
\begin{equation*}
\int_{\mathfrak{m}}\left|\mathcal{F}_{k}(\alpha)\right|^{2 s} \ll Y^{2 s-k+\epsilon} \max \left\{\frac{1}{Y}, \frac{1}{P}\right\} \tag{2.2}
\end{equation*}
$$

Based on Lemma 2.6, the following lemma is valid by interpolation:
Lemma 2.7 ([|Wo1], Lemma 3.1; [吅], Theorem 11). Assume the standard choice of $\mathcal{M}$ and $\mathfrak{m}$. Suppose that $s>B(k)$, then we have

$$
\int_{\mathfrak{m}}\left|\mathscr{F}_{k}(\alpha)\right|^{s} d \alpha \ll X^{\frac{s}{k}-1-\delta+\epsilon}
$$

for some $\delta>0$ depending on $s$ and any $\epsilon>0$.
By Equation 2.2, we can improve Lemma 2.7 slightly. That is, if we choose $P=$ $X^{1 / k-\eta}$ for sufficiently small $\eta$, we will establish the power saving for the same $s$. This will help to prove the remark at the end of Section 4.

### 2.2 Some Lemmas on $\mathcal{G}_{f}(\alpha)$ and $\mathcal{G}_{\pi}(\alpha)$

For $f$ either a holomorphic or Maass cusp form of $\mathrm{SL}_{2}(\mathbb{Z})$, the Hecke bound is:

Lemma 2.8 (Hecke' bound, [โw1], [Iw2]). Let $a_{f}(n)$ be the $n$-th normalized Fourier coefficient of $f$. Then for any $\alpha \in \mathbb{R}$,

$$
\sum_{n \leq X} a_{f}(n) e(\alpha n) \ll_{f, \epsilon} X^{\frac{1}{2}+\epsilon}
$$

with any $\epsilon>0$ and this estimate is uniformly for $\alpha \in[0,1)$.
Then we have the Voronoi formula for $f(z)$, either a holomorphic or Maass cusp form.
Lemma 2.9 ([|Go]). Let $f(z)=\sum_{n \geq 1} a_{f}(n) n^{(l-1) / 2} e(n z) \in \mathcal{S}_{l}\left(\mathrm{SL}_{2}(\mathbb{Z})\right)$. Let $\psi$ be a compactly supported function. For any integer $q \leq 1$ with $(a, q)=1$, we have:

$$
\sum_{n=1}^{\infty} a_{f}(n) e\left(\frac{a n}{q}\right) \psi(n)=\frac{2 \pi}{q} i^{l} \sum_{n=1}^{\infty} a_{f}(n) e\left(-\frac{\bar{a} n}{q}\right) \Phi\left(\frac{n}{q^{2}}\right)
$$

where $\bar{a}$ is the multiplicative inverse of $a(\bmod q)$ and

$$
\Phi(y)=\int_{0}^{\infty} \psi(x) J_{l-1}(4 \pi \sqrt{y x}) d x
$$

Lemma $2.10([\overline{\mathrm{Me}}])$. Let $f(z)=y^{1 / 2} \sum_{n \neq 0} a_{f}(n) K_{i t}(2 \pi|n| y) e(n x)$ be a Maass cusp form for $\mathrm{SL}_{2}(\mathbb{Z})$. Let $\psi$ be a compactly supported function. For any integer $q \leq 1$ with $(a, q)=1$, we have:

$$
\sum_{n=1}^{\infty} a_{f}(n) e\left(\frac{a n}{q}\right) \psi(n)=\frac{1}{q} \sum_{ \pm} \sum_{n=1}^{\infty} a_{f}(\mp n) e\left( \pm \frac{\bar{a} n}{q}\right) H^{ \pm}\left(\frac{n}{q^{2}}\right)
$$

where $\bar{a}$ is the multiplicative inverse of $a(\bmod q)$ and

$$
\begin{aligned}
& H^{-}(y)=-\frac{\pi}{\cosh \pi t} \int_{0}^{\infty} \psi(x)\left\{Y_{2 i t}+Y_{-2 i t}\right\}(4 \pi \sqrt{x y}) d x \\
& H^{+}(y)=4 \cosh \pi t \int_{0}^{\infty} \psi(x) K_{2 i t}(4 \pi \sqrt{x y}) d x
\end{aligned}
$$

Suppose that $\phi^{(j)}(x) \ll{ }_{j} 1$. Via integration by parts, we can show that the sums on the right hand side of Lemma 2.9 and Lemma 2.10 are essentially supported on $n \ll \frac{q^{2} X^{\epsilon}}{X}$ for any $\epsilon>0$. The contribution from the terms with $n \gg \frac{q^{2} X^{\epsilon}}{X}$ is negligibly small. For smaller values, we have the trivial bound

$$
\Phi\left(\frac{n}{q^{2}}\right), H^{ \pm}\left(\frac{n}{q^{2}}\right) \ll X
$$

To prove Theorem 1.6 , We will apply the Voronoi formula for

$$
\sum_{n=1}^{\infty} a_{\pi}(1,1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) .
$$

Here we will first introduce the general theory for the Voronoi formula of $\mathrm{GL}(m)$. Let $\pi$ be a Maass cusp form of $\mathrm{SL}_{m}(\mathbb{Z})$ and $\psi$ a smooth function compactly supported in $(0, \infty)$. Then we define

$$
G_{ \pm, \psi}(x)=\frac{1}{2 \pi i} \int_{\operatorname{Re} s=-\sigma} \widetilde{\psi}(s) x^{s} F_{ \pm}(s) d s
$$

where $\sigma>0$,

$$
\widetilde{\psi}(s)=\int_{0}^{\infty} \psi(x) x^{s} \frac{d x}{x}
$$

is the Mellin transformation of $\psi$, and

$$
F_{ \pm}(s)=\frac{1}{2} \pi^{-m(1 / 2-s)}\left(\prod_{j=1}^{m} \frac{\Gamma\left(\frac{1-s-\bar{\mu}_{\pi}(j)}{2}\right)}{\Gamma\left(\frac{s-\mu_{\pi}(j)}{2}\right)} \pm i^{-m} \frac{\Gamma\left(\frac{2-s-\bar{\mu}_{\pi}(j)}{2}\right)}{\Gamma\left(\frac{1+s-\mu_{\pi}(j)}{2}\right)}\right)
$$

with $\left\{\bar{\mu}_{\pi}(j)\right\}_{j=1}^{m}$ being the Langlands parameter for $\tilde{\pi}$, the dual of $\pi$. For $\overrightarrow{\mathbf{d}}=\left(d_{1}, \ldots, d_{m-2}\right)$ being an ( $m-2$ )-tuple of integers, we define

$$
h(\overrightarrow{\mathbf{d}})=\prod_{i=1}^{m-2} d_{i}^{m-i} .
$$

Then we have the following Voronoi formula:
Theorem 2.11 ([J]], Lemma 2.4). Let $\pi$ be an even Hecke-Maass form and $\psi$ a compactly supported smooth function on $(0, \infty)$. We have, for $(a, q)=1$,

$$
\begin{aligned}
& \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n) e\left(\frac{a n}{q}\right) \psi(n)= \\
& \quad q \sum_{ \pm} \sum_{d_{1} \mid q} \sum_{d_{2} \left\lvert\, \frac{q}{d_{1}}\right.} \ldots \sum_{d_{m-2} \left\lvert\, \frac{q}{d_{1} \cdot d_{m-3}}\right.} \sum_{n=1}^{\infty} \frac{a_{\pi}\left(n, d_{m-2} \ldots, d_{1}\right)}{d_{1} \cdots d_{m-2} n} \mathrm{KL}_{m-2}(\bar{a}, \pm n ; \overrightarrow{\mathbf{d}}, q) G_{\mp, \psi}\left(\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}\right) .
\end{aligned}
$$

where $\bar{a}$ denotes the multiplication inverse of $a(\bmod q)$ and $\operatorname{KL}_{m-2}(a, \mp n ; \bar{d}, q)$ is the hyper-Kloosterman sum:

$$
\begin{array}{r}
\mathrm{KL}_{m-2}(a, n ; \overrightarrow{\mathbf{d}}, q)=\sum_{t_{1}\left(\bmod q / d_{1}\right)}{ }^{*} e\left(\frac{a t_{1}}{q / d_{1}}\right)_{t_{2}\left(\bmod q /\left(d_{1} d_{2}\right)\right)}{ }^{*} e\left(\frac{\overline{t_{1}} t_{2}}{q /\left(d_{1} d_{2}\right)}\right) \\
\\
\cdots \sum_{t_{m-2}\left(\bmod q /\left(d_{1} \cdots d_{m-2}\right)\right)}{ }^{*} e\left(\frac{\overline{t_{m-3}} t_{m-2}+n \overline{t_{m-2}}}{q /\left(d_{1} \cdots d_{m-2}\right)}\right) .
\end{array}
$$

For simplicity, we set

$$
q_{i}=\frac{q}{d_{1} \cdots d_{i}}
$$

This gives $q_{j} \mid q_{j-1}$. Then for any positive integer $n$, we can define $n^{* *}$ to be the largest square-full divisor of $n$ and $n^{*}=\frac{n}{n^{* *}}$. Then $n^{*}$ is square-free, $n=n^{*} n^{* *}$ and $\left(n^{*}, n^{* *}\right)=1$.

We have the following several lemmas:
Lemma 2.12 ([因Y], Theorem 1.1). Let $m \geq 4,\left(n, q_{1}\right)=1$. Then

$$
\sum_{a=1}^{q_{1}}\left|\mathrm{KL}_{m-2}(a, n ; \overrightarrow{\mathbf{d}}, q)\right|^{2}= \begin{cases}\lambda_{n} q_{1} \varphi\left(q_{1}\right) q_{m-2}^{m-3} & \text { if } q_{2}^{* *}=\cdots=q_{m-2}^{* *} \\ 0 & \text { otherwise }\end{cases}
$$

where $\varphi$ is the Euler function. Here $\lambda_{n}$ satisfies $0<\lambda_{n}<1$.
Additionally, for $G_{ \pm, \psi}$, we have the following lemma in [JL]:
Lemma 2.13 ([]L], Lemma 2.7). Let $G_{ \pm, \psi}(x)$ be defined above. Suppose that

$$
\psi^{(j)} \ll X^{-j} \quad \text { for } j \geq 0 \quad \int\left|\psi^{(j)}(\xi)\right| d \xi \ll X^{-j+1} \quad \text { for } j \geq 1
$$

Then for any $A>0$,

$$
G_{ \pm, \psi}(x) \ll \begin{cases}X^{-A} & \text { if } x>X^{-1} \\ (x X)^{1 / 2} & \text { if } x \leq X^{-1}\end{cases}
$$

Remark 2.14. In the proof of Lemma 2.13 in [JL], they first showed the following result:

$$
G_{ \pm, \psi}(x) \ll_{\sigma}\left(\frac{1}{x X}\right)^{\sigma}
$$

Lemma 2.13 , together with the remark, will give the following proposition:
Proposition 2.15. Let $\epsilon>0$ be a fixed positive number and $\pi$ an even Hecke-Maass cusp form on $\mathrm{GL}_{m}(\mathbb{Z})$ with $m \geq 3$. Suppose that $q \ll P=X^{1 / m-\epsilon}$ and $(a, q)=1$, then

$$
\sum_{n \geq 1} a_{\pi}(1, \ldots, 1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \lll_{\epsilon, \pi, A} X^{-A}
$$

for any $A>0$.
Proof We can apply the Voronoi formula 2.11 and obtain

$$
\begin{aligned}
& \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) e\left(\frac{a n}{q}\right) \psi(n)= \\
& \quad q e\left(-\frac{a}{q}\right) \sum_{ \pm} \sum_{d_{1} \mid q} \sum_{d_{2} \left\lvert\, \frac{q}{d_{1}}\right.} \cdots \sum_{d_{m-2} \frac{q}{d_{1} \cdot T_{m-3}}} \sum_{n=1}^{\infty} \frac{a_{\pi}\left(n, d_{m-2} \ldots, d_{1}\right)}{d_{1} \cdots d_{m-2} n} \mathrm{KL}_{m-2}(\bar{a}, \pm n ; \overrightarrow{\mathbf{d}}, q) G_{\mp, \psi}\left(\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}\right)
\end{aligned}
$$

with $\psi(x)=\phi\left(\frac{n-1}{X}\right)$. Obviously $\psi$ satisfies the conditions in Lemma 2.13.
Assume that $q^{\frac{1}{m}-\epsilon} \ll X$. Then we have

$$
\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}>X^{-1}
$$

for any $n \geq 1$. Actually this could be improved to

$$
\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}>\frac{n^{\epsilon}}{X}
$$

Then by the remark bellow Lemma 2.13, we can show that

$$
G_{ \pm, \psi}\left(\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}\right) \ll(n X)^{-A}
$$

for any sufficiently large $A$.
Notice that the hyper-Kloosterman sum can be trivially bounded by

$$
\mathrm{KL}_{m-2}(\bar{a}, \pm n ; \overrightarrow{\mathbf{d}}, q) \leq q^{m-2}
$$

and we finish the proof.
To prove Theorem 1.6, we also need the following lemma:
Lemma 2.16 ([RY], (1.15)). Let $a_{\pi}\left(n_{1}, \ldots, n_{m-1}\right)$ be the Whittaker-Fourier coefficient of $\pi$, a cusp form of $\mathrm{SL}_{m}(\mathbb{Z})$. Then for $a_{\pi}\left(n, d_{m-2}, \ldots, d_{1}\right)$, we have

$$
\sum_{n h(\overrightarrow{\mathbf{d}}) \leq X}\left|a_{\pi}\left(n, d_{m-2}, \ldots, d_{1}\right)\right|^{2} \ll X^{1+\epsilon}
$$

for any $\epsilon>0$.
If we set $d_{1}=d_{2}=\cdots=d_{m-2}=1$, and take the dual of $\pi$, Lemma 2.16 shows:
Corollary 2.17. Let $a_{\pi}\left(n_{1}, \ldots, n_{m-1}\right)$ be the Whittaker-Fourier coefficient of $\pi$, a cusp form of $\mathrm{SL}_{m}(\mathbb{Z})$. Then we have

$$
\begin{equation*}
\sum_{n \leq X}\left|a_{\pi}(1, \ldots, 1, n)\right|^{2} \ll X^{1+\epsilon} \tag{2.3}
\end{equation*}
$$

for any $\epsilon>0$. By Cauchy's inequality,

$$
\begin{equation*}
\sum_{n \leq X}\left|a_{\pi}(1, \ldots, 1, n)\right| \ll X^{1+\epsilon} \tag{2.4}
\end{equation*}
$$

## 3 Proof of Theorem 1.1

Proof of Theorem 1.1 The proof of the Maass cusp form case can be similar to that of the holomorphic cusp form case. So we only consider the case when $f$ is a holomorphic cusp form.

To prove Theorem 1.1, we use the standard circle method, that is, $P=\frac{X^{1 / k}}{2 k}$ and $Q=\frac{X}{P}$. We first consider the major arc part in Equation 2.1. By Theorem 2.1, the major arc can be written as:

$$
\frac{\Gamma\left(1+\frac{1}{k}\right)^{s}}{\Gamma\left(\frac{s}{k}\right)} \sum_{n=1}^{\infty} a_{f}(n+1) \Im_{s, k}(n, P) n^{\frac{s}{k}-1} \phi\left(\frac{n}{X}\right)+O\left(X^{\frac{s}{k}-\delta}\right)
$$

Then by inserting the definition of $\mathfrak{S}_{s, k}(n, P)$, the first term in the right hand side becomes:

$$
\sum_{q \leq P} \frac{1}{q^{s}} \sum_{a(\bmod q)}^{*} S(q, a)^{s} \sum_{n=1}^{\infty} a_{f}(n+1) n^{\frac{s}{k}-1} e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right)
$$

By Abel summation, Voronoi formula 2.9 and the fact that $q \leq X^{1 / k} \leq X^{\frac{1}{2}-\frac{1}{6}}$, this is negligibly small.

Next we consider the minor arc in Equation 2.1. Recall that

$$
A(k)= \begin{cases}\frac{k(k+2)}{4} & \text { if } k \text { even } \\ \frac{(k+1)^{2}}{4} & \text { if } k \text { odd }\end{cases}
$$

First, suppose that $k$ is even. Then by Hua's inequality 2.5. we have

$$
\begin{equation*}
\int_{0}^{1}\left|\mathscr{F}_{k}(\alpha)\right|^{A(k)} d \alpha \ll X^{\frac{k}{4}+\epsilon} \tag{3.1}
\end{equation*}
$$

Then by Weyl's inequality 2.4, we have

$$
\begin{equation*}
\max _{\alpha \in \mathfrak{m}}\left|\mathcal{F}_{k}(\alpha)\right| \ll X^{1 / k-\delta+\epsilon} \tag{3.2}
\end{equation*}
$$

for some $\delta>0$.
Then for $s>A(k)$, by Hecke's bound 2.8. Equation 3.2 and Equation 3.1, we have

$$
\begin{aligned}
\int_{\mathfrak{m}} \mathcal{G}_{f}(\alpha) \mathscr{F}_{k}(\alpha)^{s} d \alpha & \ll X^{\frac{1}{2}+\epsilon} X^{(s-A(k))\left(\frac{1}{k}-\delta+\epsilon\right)} \int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{A(k)} d \alpha \\
& \ll X^{\frac{1}{2}+\frac{s-A(k)}{k}-(s-A(k)) \delta+\frac{k}{4}+\epsilon}=X^{\frac{s}{k}-(s-A(k)) \delta+\epsilon}
\end{aligned}
$$

Next we consider the odd case. Notice that $A(k)=\frac{1}{2} \frac{(k-1)(k+1)}{4}+\frac{1}{2} \frac{(k+1)(k+3)}{4}$. Then by Cauchy-Schwarz inequality and Hua's inequality 2.5, we have

$$
\begin{equation*}
\int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{A(k)} d \alpha \leq\left(\int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{\frac{(k-1)(k+1)}{4}}\right)^{1 / 2}\left(\int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{\frac{(k+1)(k+3)}{4}}\right)^{1 / 2} \ll X^{\frac{k^{2}+1}{4 k}} \tag{3.3}
\end{equation*}
$$

Then we apply Hecke's bound 2.8, Equation 3.2 and Equation 3.3:

$$
\begin{aligned}
\int_{\mathfrak{m}} \mathcal{G}_{f}(\alpha) \mathcal{F}_{k}(\alpha)^{s} d \alpha & \ll X^{\frac{1}{2}+\epsilon} X^{(s-A(k))\left(\frac{1}{k}-\delta+\epsilon\right)} \int_{0}^{1}\left|\mathcal{F}_{k}(\alpha)\right|^{A(k)} d \alpha \\
& \ll X^{\frac{1}{2}+\frac{s-A(k)}{k}-(s-A(k)) \delta+\frac{k^{2}+1}{4 k}+\epsilon}=X^{\frac{s}{k}-(s-A(k)) \delta+\epsilon}
\end{aligned}
$$

Combine with the results on the major arcs, we obtain, for $s>A(k)$,

$$
\sum_{n=1}^{\infty} a_{f}(n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{k}-\delta+\epsilon}
$$

for some $\delta>0$ depending on $s, k$ and any $\epsilon>0$.
Remark 3.1. By a similar argument and Miller's bound [Mi] for $\mathrm{SL}_{3}(\mathbb{Z})$ Maass cusp forms, we could show that for $s>\frac{9}{16} k^{2}+O(k)$,

$$
\sum_{n=1}^{\infty} a_{\pi}(1, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \ll X^{\frac{s}{k}-\delta+\epsilon}
$$

for some $\delta$ depending on $s, k$ and any $\epsilon>0$. However, this is weaker than Theorem 1.3

## 4 Proof of Theorem 1.3

Proof of Theorem 1.3. To prove the theorem, we need to consider the following two separated cases: $k=3$, and $k \geq 5$.

The case $k=3$ : let $\pi$ be a fixed even Maass cusp form of $\mathrm{GL}_{m}(\mathbb{Z})$ with $N \geq 3$. Fix $\epsilon>0$. In this part, set $P=X^{1 / m-\epsilon}$ and $Q=\frac{X}{P}$. Then the major arc is given by

$$
\int_{\mathcal{M}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

By virtue of Theorem 2.1, one has:

$$
\int_{\mathcal{M}} \mathcal{G}(\alpha) \mathcal{F}_{3}^{s}(\alpha) d \alpha=\frac{\Gamma\left(1+\frac{1}{3}\right)^{s}}{\Gamma\left(\frac{s}{3}\right)} \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) n^{\frac{s}{3}-1} \Im_{s, 3}(n, P) \phi\left(\frac{n}{X}\right)+O\left(X^{\frac{s}{3}-\delta^{\prime}+\epsilon}\right)
$$

Here $\delta^{\prime}$ is dependent on $s$. In fact, we can check that $\delta^{\prime} \geq \frac{1}{4 m}$.
Then by inserting the definition of $\mathbb{S}_{s, 3}(n, P)$, we get:

$$
\sum_{q \leq P} \frac{1}{q^{s}} \sum_{a(\bmod q)}^{*} S(q, a)^{s} \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) n^{\frac{s}{3}-1} e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right)
$$

By Abel summation and Proposition 2.15, this is negligibly small by the choice of $P$.

Next, for the minor arc,

$$
\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

one can apply Cauchy-Schwartz inequality and Equation 2.3 to obtain:

$$
\begin{aligned}
\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha & \leq\left(\int_{\mathfrak{m}}\left|\mathcal{G}_{\pi}(\alpha)\right|^{2} d \alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{2 s} d \alpha\right)^{\frac{1}{2}} \\
& \ll X^{\frac{1}{2}+\epsilon}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{2 s} d \alpha\right)^{\frac{1}{2}}
\end{aligned}
$$

By the trivial estimation for $\mathcal{F}_{3}(\alpha)$ and Cauchy-Schwarz inequality,

$$
\begin{equation*}
\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{2 s} d \alpha=\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{2 s-10+10} d \alpha \leq X^{\frac{2 s-10}{3}}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{8} d \alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{12} d \alpha\right)^{\frac{1}{2}} \tag{4.1}
\end{equation*}
$$

By Hua's inequality,

$$
\begin{equation*}
\int_{\mathfrak{m}}\left|\mathscr{F}_{3}(\alpha)\right|^{8} d \alpha \leq \int_{0}^{1}\left|\mathscr{F}_{3}(\alpha)\right|^{8} d \alpha \ll X^{\frac{5}{3}+\epsilon} \tag{4.2}
\end{equation*}
$$

By Equation 2.2

$$
\begin{equation*}
\int_{\mathfrak{m}}\left|\mathscr{F}_{3}(\alpha)\right|^{12} d \alpha \ll \frac{X^{3+\epsilon}}{P} \tag{4.3}
\end{equation*}
$$

Combine Equation 4.1, 4.2 and 4.3 and we have

$$
\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{3}^{2 s}(\alpha) d \alpha \ll X^{\frac{s}{3}-\frac{1}{4 m}+\epsilon}
$$

for any $\epsilon>0$. Therefore,

$$
\sum_{n \geq 1} a_{\pi}(1, \ldots, 1, n+1) r_{s, 3}(n) \phi\left(\frac{n}{X}\right)=\int_{0}^{1} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{3}^{s}(\alpha) d \alpha \ll X^{\frac{s}{3}-\frac{1}{4 m}+\epsilon}
$$

The case $k \geq 5$. We use the standard choice of $P$ and $Q$, that is, $P=\frac{X^{1 / k}}{2 k}$ and $Q=\frac{X}{P}$. First, we consider the minor arc:

$$
\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha
$$

By a similar argument to the first case, we have:

$$
\int_{\mathfrak{m}} \mathcal{G}_{\pi}(\alpha) \mathcal{F}_{k}^{s}(\alpha) d \alpha \ll X^{\frac{1}{2}+\epsilon}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{k}(\alpha)\right|^{2 s} d \alpha\right)^{\frac{1}{2}}
$$

Assume that $s>B(k)$. By Lemma 2.7, we have

$$
\int_{\mathfrak{m}}\left|\mathcal{F}_{k}(\alpha)\right|^{2 s} d \alpha \ll X^{\frac{2 s}{k}-1-\delta+\epsilon}
$$

for some $\delta$ depending on $s, k$ and any $\epsilon>0$. This shows the power saving for the minor arc.

For the major arc part, we would use a similar method to the case $k=3$. By Theorem 2.1, the definition of $\Im_{s, k}(n, P)$, and Abel summation, it suffices to show that

$$
\sum_{q \leq P} \frac{1}{q^{s}} \sum_{a(\bmod q)}^{*} S(q, a)^{s} \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \ll X^{1-\delta+\epsilon}
$$

for some $\delta>0$ and $\epsilon>0$. Additionally, this $\delta$ will give the same " $\delta$ " for the saving of the major arc part.

For any fixed sufficiently small $\epsilon$, we split the whole summation into two parts: $q \ll$ $X^{\frac{1}{m}-\epsilon}$ and $X^{\frac{1}{m}-\epsilon} \ll q \leq P$. (Noice that when $k>m$, the second part vanishes by a suitable choice of $\epsilon$.)

For $q \ll X^{\frac{1}{m}-\epsilon}$, by Proposition 2.15, this is negligibly small.
Then for the second term and $s>2 k$, by Equation 2.4 and Lemma 2.3, one has,

$$
\begin{gathered}
\sum_{X^{\frac{1}{m}-\epsilon} \ll q \leq P} \frac{1}{q^{s}} \sum_{a(\bmod q)} S(q, a)^{s} \sum_{n=1}^{\infty} a_{\pi}(1, \ldots, 1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \\
X^{\frac{1}{m}-\epsilon \ll q \leq P} \sum^{q^{s}} \sum_{a(\bmod q)}|S(q, a)|^{s} X^{1+\epsilon} \\
\ll X^{1+\epsilon} \sum_{X^{\frac{1}{m}-\epsilon} \ll q \leq P} \frac{1}{q^{\frac{s}{k}-1}}<
\end{gathered}
$$

Notice that when $k \geq 5, B(k) \geq 2 k$. Therefore, when $m \geq 3, k \geq 5$ and $s>B(k)$, one has:

$$
\sum_{n \geq 1} a_{\pi}(1, \ldots, 1, n+1) r_{s, k}(n) \phi\left(\frac{n}{X}\right) \ll_{\epsilon} X^{\frac{s}{k}-\delta+\epsilon}
$$

for some $\delta>0$ and any $\epsilon>0$.
Remark 4.1. By the virtual of second part, the power saving for $k=4$ and $m \geq 4$ will be obtained when $s>8$. That is because, $8>B(4)$. However, for $k=m=4$ and $s=8$, the power saving can also be shown by setting $P=X^{1 / 4-\eta}$ and making use of the remark bellow Lemma 2.7

## 5 Proof of Theorem 1.6

Proof of Theorem 1.6 When $m=4, k=3$ and $s=5$, we consider the standard choice, that is, $P=\frac{X^{1 / 3}}{6}$ and $Q=\frac{X}{P}$. Let $\pi$ be an even Hecke-Maass cusp form of $\mathrm{SL}_{4}(\mathbb{Z})$. Then

$$
\sum_{n \geq 1} a_{\pi}(1,1, n+1) r_{5,3}(n) \phi\left(\frac{n}{X}\right)=\int_{\mathcal{M}} \mathcal{G}(\alpha) \mathcal{F}_{3}^{5}(\alpha) d \alpha+\int_{\mathfrak{m}} \mathcal{G}(\alpha) \mathcal{F}_{3}^{5}(\alpha) d \alpha
$$

We would consider the minor arc. By Cauchy-Schwartz inequality and Equation 2.3, we have

$$
\int_{\mathfrak{m}} \mathcal{G}(\alpha) \mathcal{F}_{3}^{5}(\alpha) d \alpha<\leq\left(\int_{\mathfrak{m}}|\mathcal{G}(\alpha)|^{2} d \alpha\right)^{1 / 2}\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{10} d \alpha\right)^{1 / 2}
$$

For the second term on the right hand side, by Cauchy-Schwartz inequality, Hua's inequality and Lemma 2.6, we have

$$
\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{10} d \alpha \ll\left(\int_{\mathfrak{m}}\left|\mathcal{F}_{3}(\alpha)\right|^{8} d \alpha\right)^{1 / 2}\left(\int_{\mathfrak{M}}\left|\mathcal{F}_{3}(\alpha)\right|^{12} d \alpha\right)^{1 / 2} \ll X^{\frac{13}{6}+\epsilon}
$$

for any $\epsilon>0$. Combine these results and we obtian

$$
\int_{\mathfrak{m}} \mathcal{G}(\alpha) \mathcal{F}_{3}^{5}(\alpha) d \alpha \ll X^{\frac{5}{3}-\frac{1}{12}+\epsilon}
$$

for any $\epsilon>0$.
Next, we consider the major arc. By a argument similar to the proof of Theorem 1.3 , it suffices to show

$$
\sum_{q \leq P} \frac{1}{q^{5}} \sum_{a(\bmod q)}^{*} S(q, a)^{5} \sum_{n=1}^{\infty} a_{\pi}(1,, 1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \ll X^{1-\frac{1}{12}+\epsilon}
$$

for any $\epsilon$. Here we should point out that, when using Theorem 2.1, we need to check the error term. Fortunately, the power saving of the error term is exactly $\frac{1}{12}$.

Here for simplicity, we set

$$
\mathcal{A}(q)=\sum_{a(\bmod q)}{ }^{*} S(q, a)^{5} \sum_{n=1}^{\infty} a_{\pi}(1,, 1, n) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) .
$$

Then we apply the Voronoi formula 2.11 this gives:

$$
\begin{aligned}
\mathcal{A}(q) & =\sum_{a(\bmod q)}{ }^{*} S(q, a)^{5} \sum_{n=1}^{\infty} a_{\pi}(1,, 1, n) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \\
& =q \sum_{ \pm} \sum_{d_{1} \mid q} \sum_{d_{2} \left\lvert\, \frac{q}{d_{1}}\right.} \sum_{n=1}^{\infty} \frac{a_{\pi}\left(n, d_{2}, d_{1}\right)}{d_{1} d_{2} n} G_{\mp, \psi}\left(\frac{n h(\overrightarrow{\mathbf{d}})}{q^{m}}\right) \sum_{a(\bmod q)}{ }^{*} S(q, a)^{5} e\left(\frac{-a}{q}\right) \mathrm{KL}_{2}(-\bar{a}, \pm n ; \overrightarrow{\mathbf{d}}, q) .
\end{aligned}
$$

In this case, set

$$
\begin{equation*}
\mathcal{T}(a, q ; \pm n, \overrightarrow{\mathbf{d}})=\sum_{a(\bmod q)}{ }^{*} S(q, a)^{5} e\left(\frac{-a}{q}\right) \mathrm{KL}_{2}(-\bar{a}, \pm n ; \overrightarrow{\mathbf{d}}, q) \tag{5.1}
\end{equation*}
$$

Notice that $\mathrm{KL}_{2}\left(a+q_{1}, n ; \overrightarrow{\mathbf{d}}, q\right)=\mathrm{KL}_{2}(a, n ; \overrightarrow{\mathbf{d}}, q)$. By Lemma 2.3. Cauchy's inequality and Lemma 2.12, we have

$$
\begin{aligned}
\mathcal{T}(a, q ; n, \overrightarrow{\mathbf{d}}) & \leq \sum_{a(\bmod q)}{ }^{*}|S(q, a)|^{5}\left|\mathrm{KL}_{2}(-\bar{a}, n ; \overrightarrow{\mathbf{d}}, q)\right| \\
& \ll q^{10 / 3}\left(\sum_{a(\bmod q)} 1\right)^{1 / 2}\left(\sum_{a(\bmod q)}\left|\operatorname{KL}_{2}(a, n ; \overrightarrow{\mathbf{d}}, q)\right|^{2}\right)^{1 / 2} \\
& \ll q^{\frac{10}{3}+2+\epsilon} \frac{1}{d_{1} d_{2}^{1 / 2}} .
\end{aligned}
$$

Remark 5.1. Here notice that we can only apply Lemma 2.12 when $\left(n, q_{1}\right)=1$. However, by following the proof of Theorem 1.1 in [RY], we can find

$$
\sum_{a\left(\bmod q_{1}\right)}\left|\mathrm{KL}_{2}(a, n ; \overrightarrow{\mathbf{d}}, q)\right|^{2}=q_{1} \varphi\left(q_{1}\right) \phi\left(q_{2}\right) T\left(q_{2}, \frac{q_{2}}{\left(n, q_{2}\right)} ; 1,1\right)
$$

The definition of $T(s, r ; b, d)$ can be found in Lemma 2.1 in [RY]. Additionally, by Lemma 2.4 in [RY], we have:

$$
T\left(q_{2}, \frac{q_{2}}{\left(n, q_{2}\right)} ; 1,1\right) \ll q_{2}^{\epsilon}
$$

for any $\epsilon>0$. Therefore,

$$
\sum_{a\left(\bmod q_{1}\right)}\left|\mathrm{KL}_{2}(a, n ; \overrightarrow{\mathbf{d}}, q)\right|^{2} \ll q^{\epsilon} q_{1} q_{2} \varphi\left(q_{1}\right)
$$

Then by Lemma 2.13, we only need to consider the summation of $n$ over $1 \leq n \ll$ $\frac{q^{4}}{h(\overrightarrow{\mathbf{d}}) X}$. Then by the second part of Lemma 2.13. Cauchy's inequality and Lemma 2.16 , we have:

$$
\begin{aligned}
\mathcal{A}(q) & \ll q^{\frac{10}{3}+3+\epsilon} \sum_{ \pm} \sum_{d_{1} \mid q} \sum_{d_{2} \left\lvert\, \frac{q}{d_{1}}\right.} \sum_{n h(\overrightarrow{\mathbf{d}}) \ll \frac{q^{4}}{X}} \frac{\left|a_{\pi}\left(n, d_{2}, d_{1}\right)\right|}{d_{1} d_{2} n}\left(\frac{n h(\overrightarrow{\mathbf{d}}) X}{q^{4}}\right)^{\frac{1}{2}} \frac{1}{d_{1} d_{2}^{1 / 2}} \\
& \left.\left.\ll X^{1 / 2} q^{\frac{10}{3}+3-2+\epsilon} \sum_{ \pm} \sum_{d_{1} \mid q} \sum_{d_{2} \left\lvert\, \frac{q}{d_{1}}\right.} \frac{1}{\sqrt{d_{1} d_{2}}} \sum_{n h(\overrightarrow{\mathbf{d}}) \ll \frac{q^{4}}{X}}\left|a_{\pi}\left(n, d_{2}, d_{1}\right)\right|^{2}\right)^{1 / 2} \sum_{n h(\overrightarrow{\mathbf{d}}) \lll \frac{q^{4}}{X}} \frac{1}{n}\right)^{1 / 2} \\
& \ll q^{\frac{19}{3}+\epsilon} .
\end{aligned}
$$

When $k=3, P=\frac{X^{1 / 3}}{6}$. This shows:

$$
\sum_{q \leq P} \frac{1}{q^{5}} \sum_{a(\bmod q)}^{*} S(q, a)^{5} \sum_{n=1}^{\infty} a_{\pi}(1,, 1, n+1) e\left(-\frac{a n}{q}\right) \phi\left(\frac{n}{X}\right) \ll P^{\frac{7}{3}+\epsilon} \ll X^{\frac{7}{9}+\epsilon}
$$

Thus we show the power saving of $\frac{2}{9}$ which is greater than $\frac{1}{12}$.

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[^0]:    ${ }^{1}$ Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

